December 13, 1888.

Professor G. G. STOKES, D.C.L., President, in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

I. "Spectrum Analysis of Cadmium." By A. GRÜNWALD, Professor of Mathematics in the Imp. Roy. German Polytechnic University at Prague. Communicated by Professor LIVEING, F.R.S. Received November 26, 1888.

[Publication deferred.]

II. "On the Bending and Vibration of thin elastic Shells, especially of Cylindrical Form." By LORD RAYLEIGH, M.A., D.C.L., Sec. R.S. Received December 1, 1888.

In a former publication* "On the Infinitesimal Bending of Surfaces of Revolution," I have applied the theory of bending to explain the deformation and vibration of thin elastic shells, which are symmetrical about an axis, and have worked out in detail the case where the shell is a portion of a sphere. The validity of this application depends entirely upon the principle that when the shell is thin enough and is vibrating in one of the graver possible modes, the middle surface behaves as if it were inextensible. "When a thin sheet of matter is subjected to stress, the force which it opposes to extension is great in comparison with that which it opposes to bending. Under ordinary circumstances, the deformation takes place approximately as if the sheet were inextensible as a whole, a condition which, in a remarkable degree, facilitates calculation, though (it need scarcely be said) even bending implies an extension of all but the central layers." If we fix our attention upon one of the terms involving sines or cosines of multiples of the longitude, into which, according to Fourier's theorem, the whole deformation may be resolved, the condition of inextensibility is almost enough to define the type. If there are two edges, *e.g.*, parallel to circles of lati-

* 'London Math. Soc. Proc.,' vol. 13, p. 4, November, 1881.

tude, the solution contains two arbitrary constants; but if a pole be included, as when the shell is in the form of a hemisphere, one of the constants vanishes, and the type of deformation is wholly determined, without regard to any other mechanical condition, to be satisfied at the edge or elsewhere. It will be convenient to restate, analytically, the type of deformation arrived at [equation (5)]. If the point upon the middle surface, whose coordinates were originally a, θ, ϕ, moves to $a+\delta r$, $\theta+\delta\theta$, $\phi+\delta\phi$, the solution is

$$\left.\begin{aligned} \delta\phi &= \mathrm{A}\tan^s\tfrac{1}{2}\theta\cos s\phi \\ \delta\theta &= -\mathrm{A}\sin\theta\tan^s\tfrac{1}{2}\theta\sin s\phi \\ \delta r &= \mathrm{A}a\,(s+\cos\theta)\tan^s\tfrac{1}{2}\theta\sin s\phi \end{aligned}\right\} \ldots\ldots\ldots\ldots(1),$$

θ being the colatitude measured from the pole through which the shell is complete. Any integral value higher than unity is admissible for s. The value 0 and 1 correspond to displacements not involving strain.

In a recent paper* Mr. Love dissents from the general principle involved in the theory above briefly sketched, and rejects the special solutions founded upon it as inapplicable to the vibration of thin shells. The argument upon which I proceeded in my former paper, and which still seems to me valid, may be put thus: It is a general mechanical principle† that, if given displacements (not sufficient by themselves to determine the configuration) be produced in a system originally in equilibrium by forces of corresponding types, the resulting deformation is determined by the condition that the potential energy of deformation shall be as small as possible. Apply this to an elastic shell, the given displacements being such as not of themselves to involve a stretching of the middle surface.‡ The resulting deformation will, in general, include both stretching and bending, and any expression for the energy will contain corresponding terms proportional to the first and third powers respectively of the thickness. This energy is to be as small as possible. Hence, when the thickness is diminished without limit, the actual displacement will be one of pure bending, if such there be, consistent with the given conditions. Otherwise the energy would be of the first order (in thickness) instead of, as it might be, of the third order, in violation of the principle.

It will be seen that this argument takes no account of special conditions to be satisfied at the edge of the shell. This is the point at which Mr. Love concentrates his objections. He considers that

* "On the small free Vibrations and Deformation of a thin elastic Shell," 'Phil. Trans.,' A, 1888.

† 'Phil. Mag.,' March, 1875; 'Theory of Sound,' § 74.

‡ There are cases where no displacement (involving strain at all) is possible without stretching of the middle surface, *e.g.*, that of the complete sphere.

the general condition necessary to be satisfied at a free edge is in fact violated by such a deformation as (1). But the condition in question* contains terms proportional to the first and to the third powers respectively of the thickness, the coefficients of the former involving as factors the extensions and shear of the middle surface. It appears to me that when the thickness is diminished without limit, the fulfilment of the boundary condition requires only that the middle surface be unstretched, precisely the requirement satisfied by solutions such as (1).

Of course, so long as the thickness is finite, the forces in operation will entail some stretching of the middle surface, and the amount of this stretching will depend on circumstances. A good example is afforded by a circular cylinder with plane edges perpendicular to the axis. Let normal forces locally applied at the extremities of one diameter of the central section cause a given shortening of that diameter. That the potential energy may be a minimum, the deformation must assume more and more the character of mere bending as the thickness is reduced. The only kind of bending that can occur in this case is the purely cylindrical one in which every normal section is similarly deformed, and then the potential energy is proportional to the total length of the cylinder. We see, therefore, that if the cylinder be very long, the energy of bending corresponding to the given local contraction of the central diameter may become very great, and a heavy strain is thrown upon the principle that the deformation of minimum energy is one of pure bending.

If the small thickness of the shell be regarded as given, a point will at last be attained when the energy can be made least by a sensible local stretching of the middle surface such as will dispense with the uniform bending otherwise necessary over so great a length. But even in this extreme case it seems correct to say that, when the thickness is *sufficiently* reduced, the deformation tends to become one of pure bending.

At first sight it may appear strange that of two terms in an expression of the potential energy, the one proportional to the cube of the thickness is to be retained, while that proportional to the first power may be omitted. The fact, however, is that the large potential energy which would accompany any stretching of the middle surface is the very reason why such stretching will not occur. The comparative largeness of the coefficient (proportional to the first power of the thickness) is more than neutralised by the smallness of the stretching itself, to the *square* of which the energy is proportional.

In general, if ψ_1 be the coordinate measuring the violation of the tie which is supposed to be more and more insisted upon by increasing

* See his equation (33).

stiffness, and if the other coordinates be suitably chosen, the potential energy of the system may be expressed

$$V = \tfrac{1}{2}c_1\psi_1^2 + \tfrac{1}{2}c_2\psi_2^2 + \tfrac{1}{2}c_3\psi_3^2 + \ldots\ldots$$

This follows from the general theorem that V and T may always be reduced to sums of squares simply, if we suppose that $T = \tfrac{1}{2}a_1\dot{\psi}_1^2$.

The equations of equilibrium under the action of external forces Ψ_1, Ψ_2, .. are thus

$$\Psi_1 = c_1\psi_1, \qquad \Psi_2 = c_2\psi_2, \ldots\ldots;$$

hence if the forces are regarded as given, the effect of increasing c_1 without limit is not merely to annul ψ_1, but also the term in V which depends upon it.

An example might be taken from the case of a rod clamped at one end A, and deflected by a lateral force, whose stiffness from the end A up to a neighbouring place B, is conceived to increase indefinitely. In the limit we may regard the rod as clamped at B, and neglect the energy of the part AB, in spite of, or rather in consequence of, its infinite stiffness.

If it be admitted that the deformations to be considered are pure bendings, the next step is the calculation of the potential energy corresponding thereto. In my former paper, the only case for which this part of the problem was attempted was that of the sphere. After bending, "the principal curvatures differ from the original curvature of the sphere in opposite directions, and to an equal amount,* and the potential energy of bending corresponding to any element of the surface is proportional to the square of this excess or defect of curvature, without regard to the direction of the principal planes." Though he agrees with my conclusions, Mr. Love appears to regard the argument as insufficient. But clearly in the case of a given spherical shell, there are no other elements upon which the energy of bending could depend. "Thus the energy corresponding to the element of surface $a^2 \sin\theta \, d\theta \, d\phi$ may be denoted by

$$a^2 H(\delta\rho^{-1})^2 \sin\theta \, d\theta \, d\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where H depends upon the material and upon the thickness."

By the nature of the case H is proportional to the elastic constants and to the cube of the thickness, from which it follows by the method of dimensions that it is independent of a, the radius of the sphere.

* This is in virtue of Gauss's theorem that the product of the principal curvatures is unaffected by bending.

I did not, at the time, attempt the further determination of H, not needing it for my immediate purpose. Mr. Love has shown that

$$\mathrm{H} = \tfrac{4}{3} n h^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where $2h$ represents the thickness, and n is the constant of rigidity. Why n alone should occur, to the exclusion of the constant of compressibility, will presently appear more clearly.

The application of (2) to the displacements expressed in (1) gave [equation (18)]

$$\mathrm{V} = 2\pi\Sigma\,(s^3-s)\,\mathrm{A}_s^2 \int_0^\theta \mathrm{H}\,\sin^{-3}\theta\,\tan^{2s}\tfrac{1}{2}\theta\,d\theta \quad \ldots\ldots\ldots (4),$$

θ being the colatitude of the (circular) edge. In the case of the hemisphere of uniform thickness

$$\mathrm{V} = \tfrac{1}{2}\pi\mathrm{H}\Sigma\,(s^3-s)\,(2s^2-1)\,\mathrm{A}_s^2 \quad \ldots\ldots\ldots\ldots\ldots (5).$$

The calculation of the pitch of free vibration then presented no difficulty. If σ denote the superficial density, and $\cos pt$ represent the type of vibration, p_2 corresponding to $s = 2$, p_3 to $s = 3$, and so on, it appeared that

$$p_2 = \frac{\sqrt{\mathrm{H}}}{a^2\sigma} \times 5{\cdot}2400, \qquad p_3 = \frac{\sqrt{\mathrm{H}}}{a^2\sigma} \times 14{\cdot}726, \qquad p_4 = \frac{\sqrt{\mathrm{H}}}{a^2\sigma} \times 28{\cdot}462\,;$$

so that

$$p_3/p_2 = 2{\cdot}8102, \qquad p_4/p_3 = 5{\cdot}4316,$$

determining the *intervals* between the graver notes.

If the form of the shell be other than spherical, the middle surface is no longer symmetrical with respect to the normal at any point, and the expression of the potential energy is more complicated. The question is now not merely one of the curvature of the deformed surface; account must also be taken of the correspondence of normal sections before and after deformation.* A complete investigation has been given by Love; but the treatment of the question now to be explained, even if less rigorous, may help to throw light upon this somewhat difficult subject.

In the actual deformation of a material sheet of finite extent there will usually be at any point not merely a displacement of the point itself, but a rotation of the neighbouring parts of the sheet, such as a

* An extreme case may serve as an illustration. Suppose that the bending is such that the principal planes retain their positions relatively to the material surface, but that the principal curvatures are exchanged. The nature of the curvature at the point in question is the same after deformation as before, and by a rotation through 90° round the normal the surfaces may be made to fit; nevertheless the energy of bending is finite.

rigid body may undergo. All this contributes nothing to the energy. In order to take the question in its simplest form, let us refer the original surface to the normal and principal tangents at the point in question as axes of coordinates, and let us suppose that after deformation, the lines in the sheet originally coincident with the principal tangents are brought back (if necessary) to occupy the same positions as at first. The possibility of this will be apparent when it is remembered that in virtue of the inextensibility of the sheet, the angles of intersection of all lines traced upon it remain unaltered. The equation of the original surface in the neighbourhood of the point being

$$z = \tfrac{1}{2}\left(\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6),$$

that of the deformed surface may be written

$$z = \tfrac{1}{2}\left\{\frac{x^2}{\rho_1 + \delta\rho_1} + \frac{y^2}{\rho_2 + \delta\rho_2} + 2\tau xy\right\} \quad \ldots\ldots\ldots\ldots \quad (7).$$

In strictness $(\rho_1 + \delta\rho_1)^{-1}$, $(\rho_2 + \delta\rho_2)^{-1}$ are the curvatures of the sections made by the planes $x = 0$, $y = 0$; but since principal curvatures are a maximum or a minimum, they represent with sufficient accuracy the new principal curvatures, although these are to be found in slightly different planes. The condition of inextensibility shows that points which have the same x and y in (6) and (7) are *corresponding* points, and by Gauss's theorem it is further necessary that

$$\frac{\delta\rho_1}{\rho_1} + \frac{\delta\rho_2}{\rho_2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8).$$

It thus appears that the energy of bending will depend upon two quantities, one giving the alterations of principal curvature, and the other τ depending upon the shift (in the material) of the principal planes.

In calculating the energy we may regard it as due to the stretchings and contractions under tangential forces of the various infinitely thin laminæ into which the shell may be divided. The middle lamina being unstretched, makes no contribution. Of the other laminæ, the stretching is in proportion to the distance from the middle surface, and the energy of stretching is therefore as the square of this distance. When the integration over the whole thickness of the shell is carried out, the result is accordingly proportional to the cube of the thickness.

The next step is to estimate more precisely the energy corresponding to a small element of area of a lamina. The general equations in

three dimensions, as given in Thomson and Tait's 'Natural Philosophy,' § 694, are

$$na = \mathrm{S}, \qquad nb = \mathrm{T}, \qquad nc = \mathrm{U} \quad \ldots\ldots\ldots (9)$$

$$\left.\begin{aligned} \mathrm{M}e &= \mathrm{P}-\sigma\,(\mathrm{Q}+\mathrm{R}) \\ \mathrm{M}f &= \mathrm{Q}-\sigma\,(\mathrm{R}+\mathrm{P}) \\ \mathrm{M}g &= \mathrm{R}-\sigma\,(\mathrm{P}+\mathrm{Q}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (10),$$

where

$$\rho = \frac{m-n}{2m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11).*$$

The energy w, corresponding to the unit of volume, is given by

$$2w = (m+n)\,(e^2+f^2+g^2)$$
$$+2\,(m-n)\,(fg+ge+ef)+n\,(a^2+b^2+c^2) \quad \ldots\ldots (12).$$

In the application to a lamina, supposed parallel to xy, we are to take $\mathrm{R} = 0$, $\mathrm{S} = 0$, $\mathrm{T} = 0$; so that

$$g = -\sigma\,\frac{e+f}{1-\sigma}, \qquad a = 0, \qquad b = 0.$$

Thus in terms of the elongations e, f, parallel to x, y, and of the shear c, we get

$$w = n\left\{e^2+f^2+\frac{m-n}{m+n}\,(e+f)^2+\tfrac{1}{2}\,c^2\right\} \quad \ldots\ldots (13).$$

We have now to express the elongations of the various laminæ of a shell when bent, and we will begin with the case where $\tau = 0$, that is, when the principal planes of curvature remain unchanged. It is evident that in this case the shear c vanishes, and we have to deal only with the elongations e and f parallel to the axes. In the section by the plane of zx, let s, s' denote corresponding infinitely small arcs of the middle surface and of a lamina distant h from it. If ψ be the angle between the terminal normals, $s = \rho_1\psi$, $s' = (\rho_1+h)\,\psi$, $s'-s = h\,\psi$. In the bending, which leaves s unchanged,

$$\delta s' = h\,\delta\psi = h\,s\,\delta\,(1/\rho_1).$$

Hence

$$e = \delta s'/s' = h\,\delta(1/\rho_1),$$

and in like manner $f = h\,\delta\,(1/\rho_2)$. Thus for the energy U per unit of *area* we have

* M is Young's modulus, σ is Poisson's ratio, n is the constant of rigidity, and $(m-\frac{1}{3}n)$ that of cubic compressibility. In terms of Lamé's constants (λ, μ), $m = \lambda+\mu$, $n = \mu$.

$$dU = nh^2dh\left\{\left(\delta\frac{1}{\rho_1}\right)^2+\left(\delta\frac{1}{\rho_2}\right)^2+\frac{m-n}{m+n}\left(\delta\frac{1}{\rho_1}+\delta\frac{1}{\rho_2}\right)^2\right\},$$

and on integration over the whole thickness of the shell ($2h$) *

$$U = \frac{2nh^3}{3}\left\{\left(\delta\frac{1}{\rho_1}\right)^2+\left(\delta\frac{1}{\rho_2}\right)^2+\frac{m-n}{m+n}\left(\delta\frac{1}{\rho_1}+\delta\frac{1}{\rho_2}\right)^2\right\} \quad \ldots(14).$$

This conclusion may be applied at once, so as to give the result applicable to a spherical shell; for, since the original principal planes are arbitrary, they can be taken so as to coincide with the principal planes after bending. Thus $\tau = 0$; and by Gauss's theorem

$$\delta\frac{1}{\rho_1}+\delta\frac{1}{\rho_2} = 0,$$

so that

$$U = \frac{4nh^3}{3}\left(\delta\frac{1}{\rho_1}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15),$$

where $\delta\rho^{-1}$ denotes the change of principal curvature. Since $e = -f$, $g = 0$, the various laminæ are simply sheared, and that in proportion to their distance from the middle surface. The energy is thus a function of the constant of rigidity only.

The result (14) is applicable directly to the plane plate; but this case is peculiar in that, on account of the infinitude of ρ_1, ρ_2 (8) is satisfied without any relation between $\delta\rho_1$ and $\delta\rho_2$. Thus for a plane plate

$$U = \frac{2nh^3}{3}\left\{\frac{1}{\rho_1^2}+\frac{1}{\rho_2^2}+\frac{m-n}{m+n}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)^2\right\} \quad \ldots\ldots \quad (16),$$

where ρ_1^{-1}, ρ_2^{-1}, are the two independent principal curvatures after bending.

We have thus far considered τ to vanish; and it remains to investigate the effect of the deformations expressed by

$$\delta z = \tau xy = \tfrac{1}{2}\tau(\xi^2-\eta^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17),$$

where ξ, η relate to new axes inclined at 45° to those of x, y. The curvatures defined by (17) are in the planes of ξ, η, equal in numerical value and opposite in sign. The elongations in these directions for

* It is here assumed that m and n are independent of h, that is, that the material is homogeneous. If we discard this restriction, we may form the conception of a shell of given thickness, whose middle surface is physically inextensible, while yet the resistance to bending is moderate. In this way we may realise the types of deformation discussed in the present paper, *without supposing the thickness to be infinitely small;* and the independence of such types upon conditions to be satisfied at a free edge is perhaps rendered more apparent.

any lamina within the thickness of the shell are $h\tau$, $-h\tau$, and the corresponding energy (as in the case of the sphere just considered) takes the form

$$U' = \frac{4nh^3\tau^2}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18).$$

This energy is to be added* to that already found in (14); and we get finally

$$U = \frac{2nh^3}{3}\left\{\left(\delta\frac{1}{\rho_1}\right)^2 + \delta\left(\frac{1}{\rho_2}\right)^2 + \frac{m-n}{m+n}\left(\delta\frac{1}{\rho_1}+\delta\frac{1}{\rho_2}\right)^2 + 2\tau^2\right\} \ldots (19),$$

as the complete expression of the energy, when the deformation is such that the middle surface is unextended. We may interpret τ by means of the angle χ, through which the principal planes are shifted; thus

$$\tau = 2\chi\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (20).$$

It will now be in our power to treat more completely a problem of great interest, viz., the deformation and vibration of a cylindrical shell. In my former paper I investigated the types of bending, but without a calculation of the corresponding energy. The results were as follows.† If the cylinder be referred to columnar coordinates z, r, ϕ, so that the displacements of a point whose equilibrium coordinates are z, a, ϕ are denoted by δz, δr, $a\delta\phi$, the equations expressing inextensibility take the form

$$\frac{d\delta z}{dz} = 0, \qquad \delta r + a\frac{d\delta\phi}{d\phi} = 0, \qquad \frac{d\delta z}{d\phi} + a^2\frac{d\delta\phi}{dz} = 0 \ldots\ldots (21),$$

from which we may deduce

$$\frac{d^2\delta\phi}{dz^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

By (22), if $\delta\phi \propto \cos s\phi$, we may take

$$a\,\delta\phi = (A_s a + B_s z)\cos s\phi \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

and then, by (21)

$$\delta r = s\,(A_s a + B_s z)\sin s\phi \ldots\ldots\ldots\ldots\ldots (24),$$

$$\delta z = -s^{-1}B_s a \sin s\phi \ldots\ldots\ldots\ldots\ldots\ldots (25).$$

* There are clearly no terms involving the products of τ with the changes of principal curvature $\delta(\rho_1^{-1})$, $\delta(\rho_2^{-2})$; for a change in the sign of τ can have no influence upon the energy of the deformation defined by (7).

† The method of investigation is similar to that employed by Jellet in his memoir ("On the Properties of Inextensible Surfaces," 'Irish Acad. Trans.,' vol. 22, 1855, p. 179), to which reference should have been made.

If the cylinder be complete, s is integral; A_s and B_s are independent constants, either of which may vanish. In the latter case the displacement is in two dimensions only.* It is unnecessary to stop to consider the demonstrations of (21), inasmuch as these equations will present themselves independently in the course of the investigations which follows.

It will be convenient to replace δz, δr, $a\delta\phi$ by single letters, which, however, it is difficult to choose so as not to violate some of the usual conventions. In conformity with Mr. Love's general notation, I will write

$$\delta z = u, \qquad a\,\delta\phi = v, \qquad \delta r = w \qquad (26).$$

The problem before us is the expression of the changes of principal curvature and shifts of principal planes at any point P (z, ϕ) of the cylinder in terms of the displacements u, v, w. As in (6), take as fixed co-ordinate axes the principal tangents and normal to the undisturbed cylinder at the point P, the axis of x being parallel to that of the cylinder, that of y tangential to the circular section, and that of ζ normal, measured inwards. If, as it will be convenient to do, we measure z and ϕ from the point P, we may express the undisturbed coordinates of a material point Q in the neighbourhood of P, by

$$x = z, \qquad y = a\phi, \qquad \zeta = \tfrac{1}{2}a\phi^2 \qquad (27).$$

During the displacement the coordinates of Q will receive the increments

$$u, \quad w\sin\phi + v\cos\phi, \quad -w\cos\phi + v\sin\phi;$$

so that after displacement

$$x = z + u, \qquad y = a\phi + w\phi + v\left(1 - \tfrac{1}{2}\phi^2\right),$$

$$\zeta = \tfrac{1}{2}a\phi^2 - w\left(1 - \tfrac{1}{2}\phi^2\right) + v\phi;$$

or if u, v, w be expanded in powers of the small quantities z, ϕ,

$$x = z + u_0 + \frac{du}{dz_0}z + \frac{du}{d\phi_0}\phi + \ldots \qquad (28).$$

$$y = a\phi + w_0\phi + v_0 + \frac{dv}{dz_0}z + \frac{dv}{d\phi_0}\phi + \ldots \qquad (29).$$

* See 'Theory of Sound,' § 233.

$$\zeta = \tfrac{1}{2}a\phi^2 - w_0 - \frac{dw}{dz_0}z - \frac{dw}{d\phi_0}\phi + v_0\phi$$

$$+\tfrac{1}{2}w_0\phi^2 - \tfrac{1}{2}\frac{d^2w}{dz_0^2}z^2 - \frac{d^2w}{dz_0 d\phi_0}z\phi - \tfrac{1}{2}\frac{d^2w}{d\phi_0^2}\phi^2$$

$$+\frac{dv}{dz_0}z\phi + \frac{dv}{d\phi_0}\phi^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30),$$

u_0, v_0, . . . being the values of u, v at the point P.

These equations give the coordinates of the various points of the deformed sheet. We have now to suppose the sheet moved as a rigid body so as to restore the position (as far as the first power of small quantities is concerned) of points infinitely near P. A purely translatory motion by which the displaced P is brought back to its original position will be expressed by the simple omission in (28), (29), (30) of the terms u_0, v_0, w_0 respectively, which are independent of z, ϕ. The effect of an arbitrary rotation is represented by the additions to x, y, ζ respectively of $y\theta_3 - \zeta\theta_2$, $\zeta\theta_1 - x\theta_3$, $x\theta_2 - y\theta_1$; where for the present purpose θ_1, θ_2, θ_3 are small quantities of the order of the deformation, the square of which is to be neglected throughout. If we make these additions to (28), &c., substituting for x, y, ζ in the terms containing θ their approximate values, we find so far as the first powers of z, ϕ

$$x = z + \frac{du}{dz_0}z + \frac{du}{d\phi_0}\phi + a\phi\theta_3,$$

$$y = a\phi + w_0\phi + \frac{dv}{dz_0}z + \frac{dv}{d\phi_0}\phi - z\theta_3,$$

$$\zeta = \frac{dw}{dz_0}z - \frac{dw}{d\phi_0}\phi + v_0\phi + z\theta_2 - a\phi\theta_1.$$

Now, since the sheet is assumed to be inextensible, it must be possible so to determine θ_1, θ_2, θ_3 that to this order $x = z$, $y = a\phi$, $\zeta = 0$.

Hence $$\frac{du}{dz_0} = 0, \qquad \frac{du}{d\phi_0} + a\theta_3 = 0,$$

$$\frac{du}{dz_0} - \theta_3 = 0, \qquad w_0 + \frac{dv}{d\phi_0} = 0,$$

$$-\frac{dw}{dz_0} + \theta_2 = 0, \qquad \frac{dw}{d\phi_0} - v_0 + a\theta_1 = 0.$$

The conditions of inextensibility are thus (if we drop the suffices as no longer required)

$$\frac{du}{dz}=0, \qquad w+\frac{dv}{d\phi}=0, \qquad \frac{du}{d\phi}+a\frac{dv}{dz}=0 \;\ldots\; (31),$$

which agree with (21).

Returning to (28), &c., as modified by the addition of the translatory and rotatory terms, we get

$$x = z + \text{terms of 2nd order in } z,\ \phi,$$

$$y = a\phi + \quad ,, \qquad ,,$$

$$\zeta = \tfrac{1}{2}a\phi^2+\tfrac{1}{2}w_0\phi^2-\tfrac{1}{2}\frac{d^2w}{dz_0^2}z^2-\frac{d^2w}{dz_0d\phi_0}z\phi$$

$$-\tfrac{1}{2}\frac{d^2w}{d\phi_0^2}\phi^2+\frac{dv}{dz_0}z\phi+\frac{dv}{d\phi_0}\phi^2;$$

or since by (31) $d^2w/dz^2=0$, and $dv/d\phi=-w$,

$$\zeta = \tfrac{1}{2}a\phi^2-\tfrac{1}{2}w_0\phi^2-\frac{d^2w}{dz_0d\phi_0}z\phi-\tfrac{1}{2}\frac{d^2w}{d\phi_0^2}\phi^2+\frac{dv}{dz_0}z\phi.$$

The equation of the deformed surface after transference is thus

$$\zeta = xy\left\{\frac{1}{a}\frac{dv}{dz_0}-\frac{1}{a}\frac{d^2w}{dz_0d\phi_0}\right\}+y^2\left\{\frac{1}{2a}-\frac{1}{2a^2}w_0-\frac{1}{2a^2}\frac{d^2w}{d\phi_0^2}\right\}\ldots\ (32).$$

Comparing with (7) we see that

$$\delta\frac{1}{\rho_1}=0, \qquad \delta\frac{1}{\rho_2}=-\frac{1}{a^2}\left(w+\frac{d^2w}{d\phi^2}\right), \qquad \tau=\frac{1}{a}\left(\frac{dv}{dz}-\frac{d^2w}{dzd\phi}\right)\ldots\ (33);$$

so that by (19)

$$U=\frac{4nh^3}{3a^2}\left\{\frac{m}{m+n}\frac{1}{a^2}\left(w+\frac{d^2w}{d\phi^2}\right)^2+\left(\frac{dv}{dz}-\frac{d^2w}{dzd\phi}\right)^2\right\}\ldots (34).$$

This is the potential energy of bending reckoned per unit of area. It can if desired be expressed by (31) entirely in terms of v.*

We will now apply (24) to calculate the whole potential energy of a complete cylinder, bounded by plane edges $z=\pm l$, and of thick-

* From Mr. Love's general equations (12), (13), (18) a concordant result may be obtained by introduction of the special conditions—

$$h_1=0, \qquad h_2=1/a, \qquad 1/\rho_1=0, \qquad 1/\rho_2=1/a,$$

limiting the problem to the case of the cylinder, and of those

$$\sigma_1=\sigma_2=\varpi=0,$$

which express the inextensibility of the middle surface.

ness which, if variable at all, is a function of z only. Since u, v, w are periodic when ϕ increases by 2π, their most general expression in accordance with (31) is [compare (23), &c.]

$$v = \Sigma[(\mathrm{A}_s a + \mathrm{B}_s z) \cos s\phi - (\mathrm{A}_s' a + \mathrm{B}_s' z) \sin s\phi] \ldots\ldots \quad (35),$$

$$w = \Sigma[s\,(\mathrm{A}_s a + \mathrm{B}_s z) \sin s\phi + s\,(\mathrm{A}_s' a + \mathrm{B}_s' z) \cos s\phi] \ldots\ldots \quad (36),$$

$$u = \Sigma[-s^{-1}\mathrm{B}_s a \sin s\phi - s^{-1}\mathrm{B}_s' a \cos s\phi] \ldots\ldots \quad (37),$$

in which the summation extends to all integral values of s from 0 to ∞. But the displacements corresponding to $s = 0$, $s = 1$ are such as a rigid body might undergo, and involve no absorption of energy. When the values of u, v, w are substituted in (34) all the terms containing products of sines or cosines with different values of s vanish in the integration with respect to ϕ, as do also those which contain $\cos s\phi \sin s\phi$. Accordingly

$$\int_0^{2\pi} \mathrm{U} a\, d\phi = \frac{4\pi n h^3}{3a} \Bigg[\frac{m}{m+n} \frac{1}{a^2} \Sigma\, (s^3 - s)^2$$

$$\{ (\mathrm{A}_s a + \mathrm{B}_s z)^2 + (\mathrm{A}_s' a + \mathrm{B}_s' z)^2 \} + \Sigma\, (s^2 - 1)^2\, (\mathrm{B}_s^2 + \mathrm{B}_s'^2) \Bigg] \ldots \quad (38).$$

Thus far we might consider h to be a function of z; but we will now treat it as a constant. In the integration with respect to z the odd powers of z will disappear, and we get as the energy of the whole cylinder of radius a, length $2l$, and thickness $2h$,

$$\mathrm{V} = \int_{-l}^{+l} \int_0^{2\pi} \mathrm{U} a\, d\phi\, dz$$

$$= \frac{8\pi n h^3 l}{3a} \Sigma\, (s^2 - 1)^2 \Bigg[\frac{m\,.\,s^2}{m+n} \Big\{ \mathrm{A}_s^2 + \mathrm{A}_s'^2$$

$$+ \frac{l^2}{3a^2} (\mathrm{B}_s^2 + \mathrm{B}_s'^2) \Big\} + \mathrm{B}_s^2 + \mathrm{B}_s'^2 \Bigg] \ldots\ldots (39),$$

in which $s = 2, 3, 4, \ldots$.

The expression (39) for the potential energy suffices for the solution of statical problems. As an example we will suppose that the cylinder is compressed along a diameter by equal forces F, applied at the points $z = z_1$, $\phi = 0$, $\phi = \pi$, although it is true that so highly localised a force hardly comes within the scope of the investigation in consequence of the stretchings of the middle surface, which will

occur in the immediate neighbourhood of the points of application.*

The work done upon the cylinder by the forces F during the hypothetical displacement indicated by δA_s, &c., will be by (36)

$$-F\Sigma\, s\,(a\delta A_s' + z_1\delta B_s')\,(1+\cos s\pi),$$

so that the equations of equilibrium are

$$\frac{dv}{dA_s} = 0, \qquad \frac{dv}{dB_s} = 0.$$

$$\frac{dv}{dA_s'} = -(1+\cos s\pi)\, saF, \qquad \frac{dv}{dB_s'} = -(1+\cos s\pi)\, sz_1F.$$

Thus for all values of s,

$$A_s = B = 0;$$

and for odd values of s,

$$A_s' = B_s' = 0.$$

But when s is even,

$$\frac{ms^2}{m+n} A_s' = -\frac{3sa^2F}{8\pi nh^3l(s^2-1)^2} \quad \ldots\ldots\ldots\ldots \quad (40),$$

$$\left\{\frac{ms^2}{m+n}\,\frac{l^2}{3a^2}+1\right\} B_s' = -\frac{3saz_1F}{8\pi nh^3l(s^2-1)^2} \quad \ldots\ldots\ldots\ldots \quad (41);$$

and the displacement w at any point (z, ϕ) is given by

$$w = 2\,(A_2'a+B_2'z)\cos 2\phi + 4\,(A_4'a+B_4'z)\cos 4\phi + \ldots\ldots\ldots\ldots \quad (42),$$

where A_2', B_2', A_4', ... are determined by (40), (41).

If the cylinder be moderately long in proportion to its diameter, the second term in the left hand member of (41) may be neglected, so that

$$\frac{l^2}{3a^2}\,\frac{B_s'}{z_1} = \frac{A_s'}{a}.$$

In this case (42) may be written

$$w = \left(1+\frac{3z_1z}{l^2}\right)\{2A_2'a\cos 2\phi + 4A_4'a\cos 4\phi + \ldots\} \ldots \quad (43),$$

* Whatever the curvature of the surface, an area upon it may be taken so small as to behave like a plane, and therefore bend, in violation of Gauss's condition, when subjected to a force which is so nearly discontinuous that it varies sensibly within the area.

showing that, except as to magnitude and sign, the curve of deformation is the same for all values of z_1 and z.*

If $z = \pm z_1$, the amplitudes are in the ratio $1 \pm 3z_1^2/l^2$; and if, further, $z_1 = l$, *i.e.*, if the force be applied at one of the ends of the cylinder, the amplitudes are as $2 : -1$. The section where the deformation (as represented by w) is zero, is given by $3zz_1 + l^2 = 0$, in which if $z_1 = l$, $z = -\frac{1}{3}l$.

When the condition as to the length of the cylinder is not imposed, the ratio $B_s' : A_s'$ is dependent upon s, and therefore the curves of deformation vary with z, apart from mere magnitude and sign. If, however, we limit ourselves to the more important term $s = 2$, we have

$$\frac{4m}{m+n}\frac{A_2'}{a} = \left\{\frac{4m}{m+n}\frac{l^2}{3a^2}+1\right\}\frac{B_2'}{z_1},$$

and $$w = 2B_2'\left\{\frac{a^2}{z_1}\left(\frac{l^2}{3a^2}+\frac{m+n}{4m}\right)+z\right\}\cos 2\phi\,;$$

so that w vanishes when

$$\frac{zz_1}{a^2}+\frac{l^2}{3a^2}+\frac{m+n}{4m} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (44).$$

This equation may be applied to find what is the length of the cylinder when the deformation just vanishes at one end if the force is applied at the other. If $z_1 = -z = l$,

$$\frac{l}{a} = \sqrt{\left\{\frac{3(m+n)}{8m}\right\}}.$$

For many materials σ [equation (11)] is about $\frac{1}{4}$, or $m = 2n$. In such cases the condition is

$$l = \tfrac{3}{4}a.$$

It should not be overlooked that although w may vanish, u remains finite.

Reverting to (23), (24), (25) we see that, if the cylinder is open at both ends, there are two types of deformation possible for each value of s. If we suppose the cylinder to be closed at $z = 0$ by a flat disk attached to it round the circumference, the inextensibility of the disk imposes the conditions, $w = \delta r = 0$, $v = a\,\delta\phi = 0$, when $z = 0$.† Hence $A_s = 0$, and the only deformation now possible is

* That w is unaltered when z and z_1 are interchanged is an example of the general law of reciprocity.

† s being greater than 1.

$$\left.\begin{aligned} v &= a\,\delta\phi = \mathrm{B}_s z \cos s\phi \\ w &= \delta r \quad = s\mathrm{B}_s z \sin s\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (45).$$

Another disk, attached where z has a finite value, would render the cylinder rigid.

Instead of a plane disk let us next suppose that the cylinder is closed at $z = 0$ by a hemisphere attached to it round the circumference. By (1) the three component displacements at the edge of the hemisphere ($\theta = \frac{1}{2}\pi$) are of the form

$$v = a\,\delta\phi = a \cos s\phi.$$

$$u = a\,\delta\theta = -a \sin s\phi.$$

$$w = \delta r \quad = sa \sin s\phi.$$

Equating these to the corresponding values for the cylinder, as given by (23), (24), (25), we get

$$\mathrm{A}_s = 1, \qquad\qquad \mathrm{B}_s = s\,;$$

so that the deformation of the cylinder is now limited to the type

$$\left.\begin{aligned} v &= (a+sz) \cos s\phi \\ w &= s\,(a+sz) \sin s\phi \\ u &= -a \sin s\phi \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (46),$$

in which we may, of course, introduce an arbitrary multiplier and an arbitrary addition to ϕ. If the convexity of the hemisphere be turned outwards, z is to be considered positive.

In like manner any other convex additions at one end of the cylinder might be treated. There are apparently three conditions to be satisfied by only two constants, but one condition is really redundant, being already secured by the inextensibility of the edges provided for in the types of deformations determined separately for the two shells. Convex additions, closing both ends of the cylinder, render it rigid, in accordance with Jellet's theorem that a closed oval shell cannot be bent.

It is of importance to notice how a cylinder, or a portion of a cylinder, can *not* be bent. Take, for example, an elongated strip, bounded by two generating lines subtending at the axis a small angle. Equations (31) [giving $d^2w/dr^2 = 0$] show that the strip cannot be bent in the plane containing the axis and the middle generating line.* The only bending symmetrical with respect to this

* This is the principle upon which metal is corrugated.

plane is a purely cylindrical one which leaves the middle generating line straight. There are two ways in which we may conceive the strip altered so as to render it susceptible of the desired kind of bending. The first is to take out the original cylindrical curvature, which reduces it to a plane strip. The second is to replace it by one in which the middle line is curved from the beginning, like the equator of a sphere or ellipsoid of revolution. In this case the total curvature being finite, the Gaussian condition can be satisfied by a change of meridianal curvature compensating the supposed change of equatorial curvature. It is easy to calculate the actual stiffness from (8) and (14), for here $\tau = 0$. We have

$$U = \frac{2nh^3}{3}\left(\delta\frac{1}{\rho_1}\right)^2\left\{1+\frac{\rho_1^2}{\rho_2^2}+\frac{m-n}{m+n}\left(1-\frac{\rho_1}{\rho_2}\right)^2\right\} \;\ldots\ldots\; (47),$$

which expresses the work per unit of area corresponding to a given bending $\delta\rho_1^{-1}$ along the equator. If $\rho_1 = \infty$, the cylindrical strip is infinitely stiff. If the curvature be spherical, $\rho_2 = \rho_1$, and

$$U = \frac{4nh^3}{3}\left(\delta\frac{1}{\rho_1}\right)^2 \;\ldots\ldots\ldots\ldots\ldots\; (48);$$

and if $\rho_2 = \infty$,

$$U = \frac{4nh^3}{3}\cdot\frac{m}{m+n}\left(\delta\frac{1}{\rho_1}\right)^2 \;\ldots\ldots\ldots\ldots\; (49).$$

Whatever the equatorial curvature may be, the ratio of stiffnesses in the two cases is equal to $m : m+n$, or about $2 : 3$, the spherically curved strip being the stiffer.

The same principle applies to the explanation of Bourdon's gauge. In this instrument there is a tube whose axis lies along an arc of a circle and whose section is elliptical, the longer axis of the ellipse being perpendicular to the general plane of the tube. If we now consider the curvature at points which lie upon the axial section, we learn from Gauss's theorem that a diminished curvature along the axis will be accompanied by a nearer approach to a circular section, and reciprocally. Since a circular form has the largest area for a given perimeter, internal pressure tends to diminish the eccentricity of the elliptic section and with it the general curvature of the tube. Thus, if one end be fixed, a pointer connected with the free end may be made to indicate the internal pressure.*

* Dec. 19.—It appears, however, that the bending of a curved tube of elliptical section cannot be pure, since the parts of the walls which lie furthest from the circular axis are necessarily stretched. The difficulty thus arising may be obviated by replacing the two halves of the ellipse, which lie on either side of the major axis, by two symmetrical curves which meet on the major axis at a *finite angle.*

We will now proceed with the calculation for the frequencies of vibration of the complete cylindrical shell of length $2l$. If the volume-density be ρ,* we have as the expression of the kinetic energy by means of (35), (36), (37).

$$\begin{aligned} \mathrm{T} &= \tfrac{1}{2} \,.\, 2h\rho \,.\, \iint (\dot{u}^2+\dot{v}^2+\dot{w}^2)\, a\, d\phi\, dz \\ &= 2\pi\rho h l a\, \Sigma\, \{a^2(1+s^2)\,(\dot{\mathrm{A}}_s+\dot{\mathrm{A}}_s'^2) \\ &\qquad + [\tfrac{1}{3}l^2(1+s^2)+s^{-2}a^2]\,(\dot{\mathrm{B}}_s^2+\dot{\mathrm{B}}_s'^2)\} \;\ldots\ldots\ldots \;(50). \end{aligned}$$

From these expressions for V and T in (39), (50) the types and frequencies of vibration can be at once deduced. The fact that the squares, and not the products, of A_s, B_s, are involved, shows that these quantities are really the principal coordinates of the vibrating system. If A_s, or A_s', vary as $\cos p_s t$, we have

$$p_s^2 = \tfrac{4}{3}\frac{mn}{m+n}\,\frac{h^2}{\rho a^4}\,\frac{(s^3-s)^2}{s^2+1} \;\ldots\ldots\ldots\ldots\ldots (51).$$

This is the equation for the frequencies of vibration in two dimensions.† For a given material, the frequency is proportional to the thickness and inversely as the square on the diameter of the cylinder.‡

In like manner if B_s, or B_s', vary as $\cos p_s't$, we find

$$p_s'^2 = \tfrac{4}{3}\,\frac{mn}{m+n}\,\frac{h^2}{\rho a^4}\,\frac{(s^3-s)^2}{s^2+1}\,\frac{1+\dfrac{3a^2}{s^2l^2}\,\dfrac{m+n}{m}}{1+\dfrac{3a^2}{(s^4+s^2)\,l^2}} \;\ldots\ldots\; (52).$$

If the cylinder be at all long in proportion to its diameter, the

According to the equations (in columnar co-ordinates) of my former paper, the conditions that δr, δz shall be independent of ϕ lead to—

$$\delta r = \mathrm{C}r, \qquad\qquad \frac{d\delta z}{dz} + \mathrm{C}\left(\frac{dr}{dz}\right)^2 = 0,$$

where C is an absolute constant.

The case where the section is a rhombus ($dr/dz = \pm \tan \alpha$) may be mentioned.

The difficulty referred to above arises when $dr/dz = \infty$.

* This can scarcely be confused with the notation for the curvature in the preceding parts of the investigation.

† See 'Theory of Sound,' § 233.

‡ There is nothing in these laws special to the cylinder. In the case of similar shells of any form, vibrating by pure bending, the frequency will be as the thicknesses and inversely as corresponding areas. If the similarity extend also to the thickness, then the frequency is inversely as the linear dimension, in accordance with the general law of Cauchy.

difference between p_s' and p_s becomes very small. Approximately in this case

$$p_s'/p_s = 1+\frac{3a^2}{2s^2l^2}\left(\frac{m+n}{m}-\frac{1}{s^2+1}\right);$$

or if we take $m = 2n$, $s = 2$,

$$p_2'/p_2 = 1+\frac{7a^2}{20l^2}.$$

In my former paper I gave the types of vibration for a circular cone, of which the cylinder may be regarded as a particular case. In terms of columnar coordinates (z, r, ϕ) we have

$$\delta\phi = (A_s+B_s z^{-1})\cos s\phi \quad \ldots\ldots\ldots\ldots \quad (53),$$

$$\delta r = s\tan\gamma\,(A_s z+B_s)\sin s\phi \quad \ldots\ldots\ldots\ldots \quad (54),$$

$$\delta z = \tan^2\gamma\,[s^{-1}B_s-s\,(A_s z+B_s)]\sin s\phi \ldots\ldots \quad (55),$$

γ being the semi-vertical angle of the cone. For the calculation of the energy of bending it would be simpler to use polar coordinates (r, θ, ϕ), r being measured from the vertex instead of from the axis.

If the cone be complete up to the vertex, we must suppose, in (53) &c., $B_s = 0$. And if we proceed to calculate the potential energy, we shall find it infinite, at least when the thickness is uniform. For since A_s is of no dimensions in length, the square of the change of curvature must be proportional to $A_s^2 z^{-2}$. When this is multiplied by $z\,dz$, and integrated, a logarithm is introduced, which assumes an infinite value when $z = 0$. The complete cone must therefore be regarded as infinitely stiff, just as the cylinder would be if one rim were held fast.

If two similar cones (bounded by circular rims) are attached so that the common rim is a plane of symmetry, the bending may be such that the common rim remains plane. If the distance of this plane from the vertex be z_1, the condition to be satisfied in (53) &c., is that $\delta z = 0$ where $z = z_1$. Hence

$$\delta\phi = A_s\left\{1-\frac{s^2}{s^2-1}\frac{z_1}{z}\right\}\cos s\phi \quad \ldots\ldots\ldots\ldots \quad (56),$$

$$\delta r = s\tan\gamma\,A_s\left\{z-\frac{s^2 z_1}{s^2-1}\right\}\sin s\phi \quad \ldots\ldots\ldots \quad (57),$$

$$\delta z = s\tan^2\gamma\,A_s\,\{z_1-z\}\sin s\phi \quad \ldots\ldots\ldots\ldots \quad (58).$$